Technische
Universität
Braunschweig

Institut für Betriebssysteme
und Rechnerverbund
Algorithmik

Algorithm Engineering
Lecture 9: SAT

SAT: Satisfiability (in Conjunctive Normal Form, CNF)

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SAT: Given

$$\varphi(x_1, \ldots, x_n) = \bigwedge_{j=1}^{k} \bigvee_{i=1}^{s_j} \ell_{ij} \text{ ; each } \ell_{ij} \in \{x_q, \overline{x_q}\} \text{ (conjunction of disjunction of literals),}$$

is there a satisfying assignment of the x_i?

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each clause is essentially a ≥ 1-constraint

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clause

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In other words: pick at least one satisfied literal from each clause.

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The original **NP-complete** problem!

Classical Backtracking & Propagation: DPLL

$\mathrm{DPLL}(X, C, \theta)$

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while C contains pure literal ℓ:

$\theta = \theta \cup \{\ell \leftarrow \texttt{true}\}$

$C = \{c_i \mid c_i \in C, \ell \notin c_i\}$

Pure Literal Elimination

We can eliminate variables that occur only positively/only negatively (pure literals) by setting them to that value; that also removes all clauses containing them.

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  return $\bot$

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As always with pseudo-code:

- Naïve implementation inefficient
- Efficient implementation non-trivial
- Need: good implementation of UP, avoid copying clauses
- Ideally, no actual recursion

DPLL: Example

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$$C = \{12\overline{3}, 23\overline{4}, 134, \overline{1}24, \overline{1}\overline{2}3, \overline{2}\overline{3}4, \overline{1}\overline{3}\overline{4}, 1\overline{2}\overline{4}\}$$

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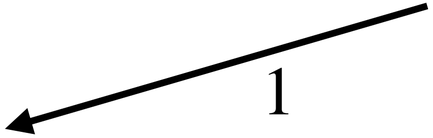

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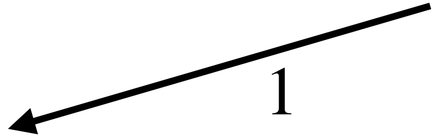

$$\{23\overline{4}, 24, \overline{2}3, \overline{2}\,\overline{3}4, \overline{3}\,\overline{4}\}$$

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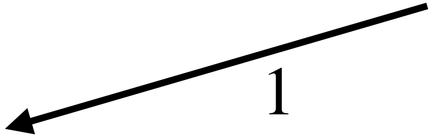

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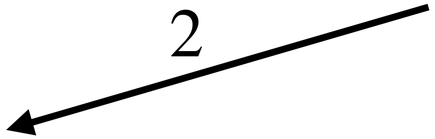

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$$\{23\overline{4}, 24, \overline{2}3, \overline{2}\overline{3}4, \overline{3}\overline{4}\}$$

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$$\{3, \overline{3}4, \overline{3}\overline{4}\}$$

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UP

$$\bot$$

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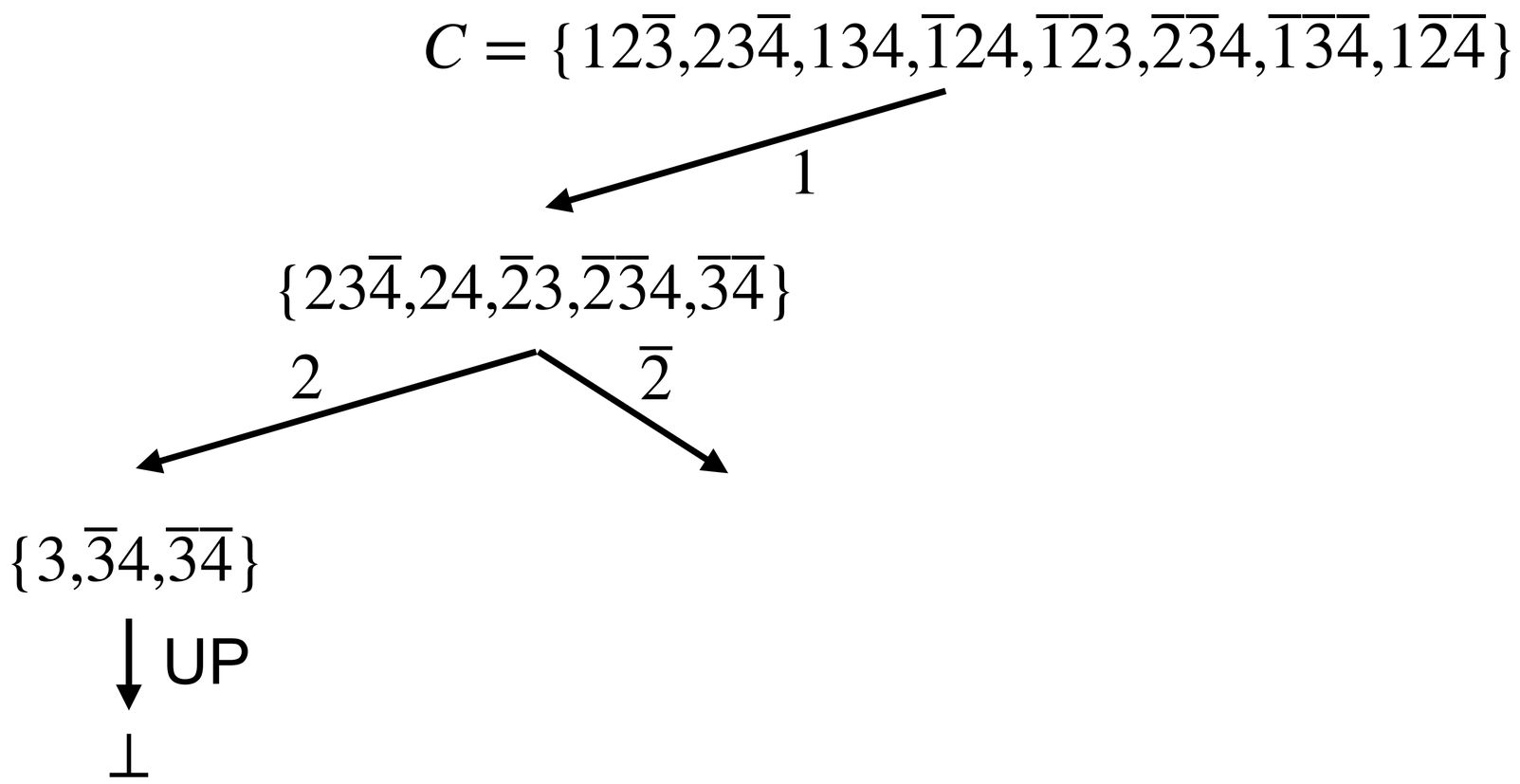

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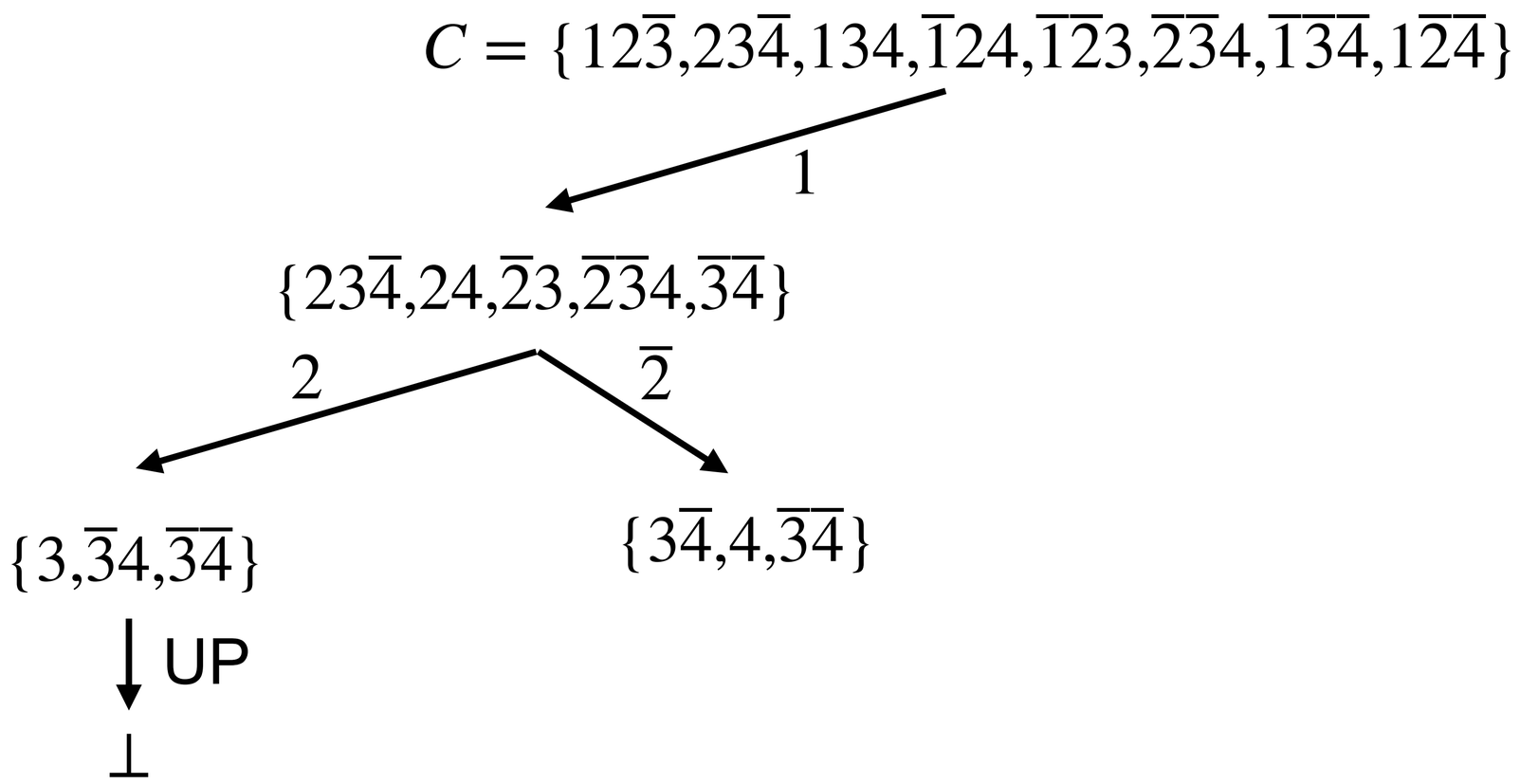

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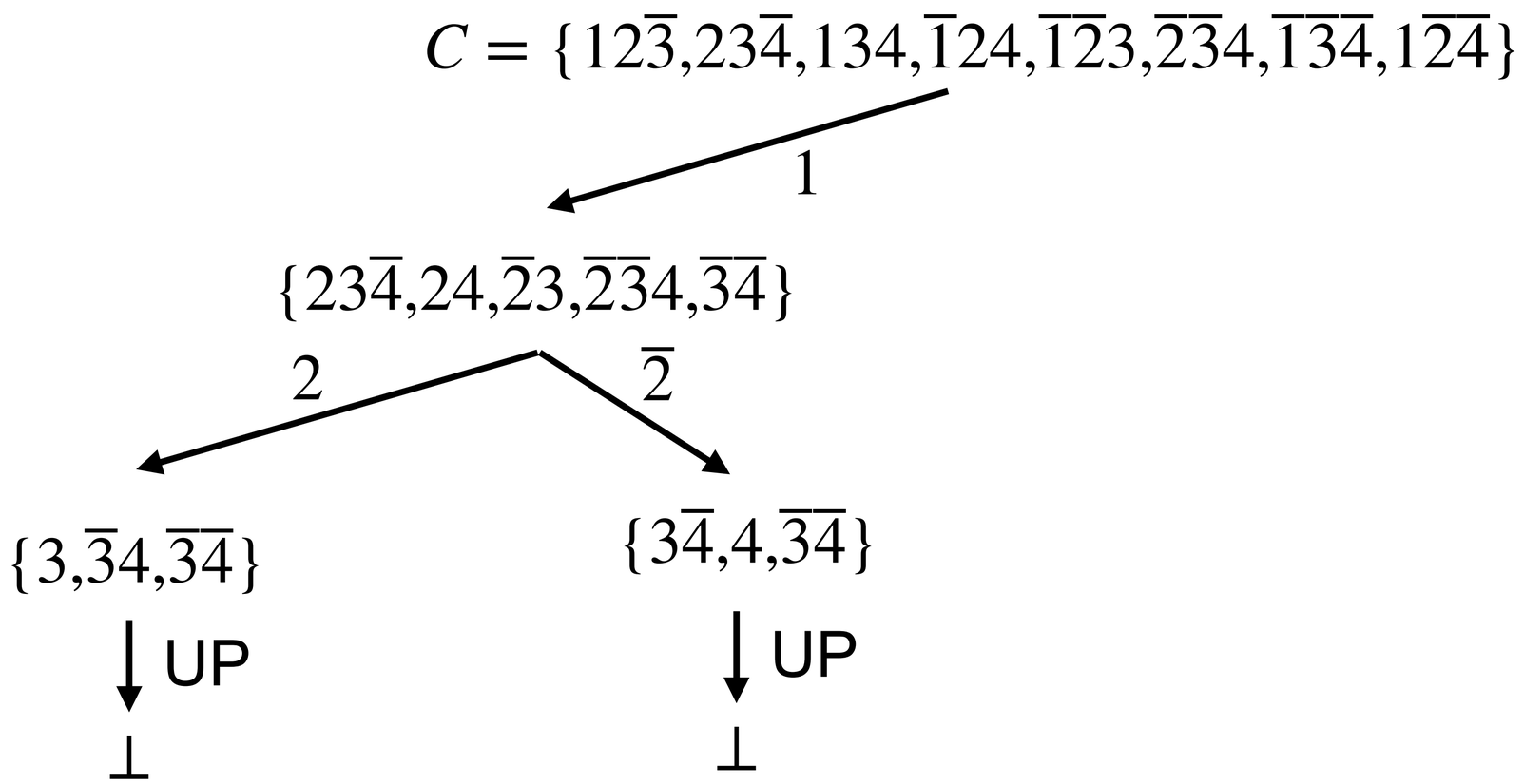

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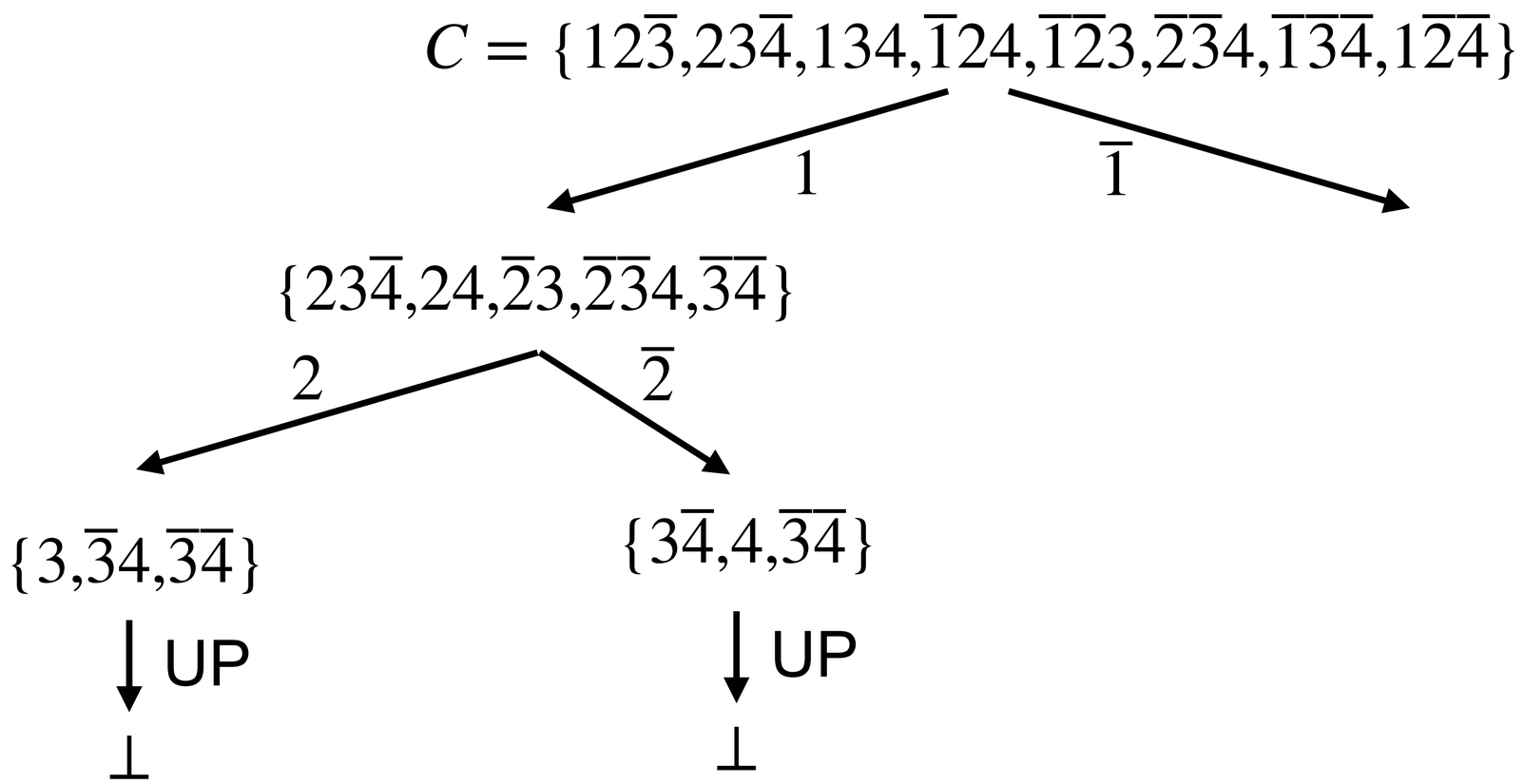

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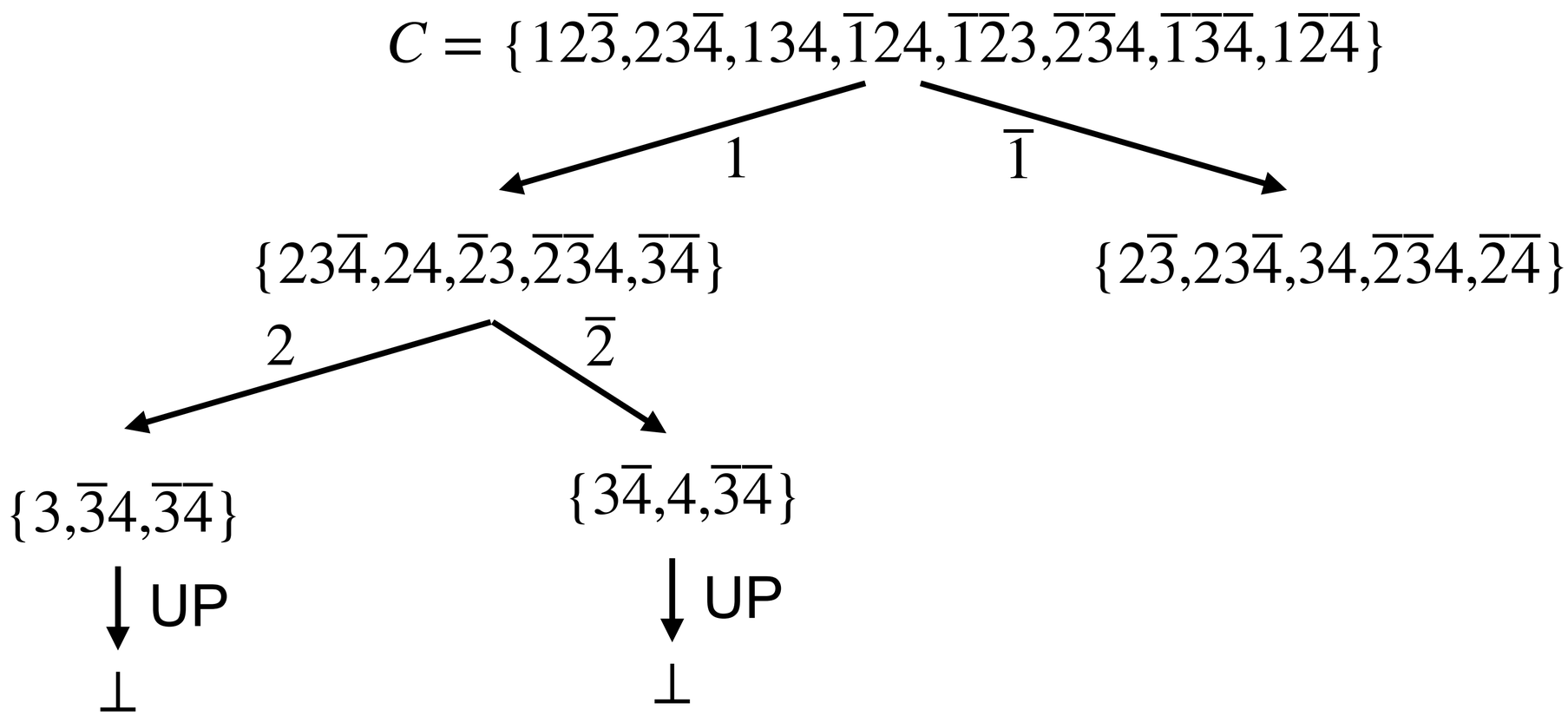

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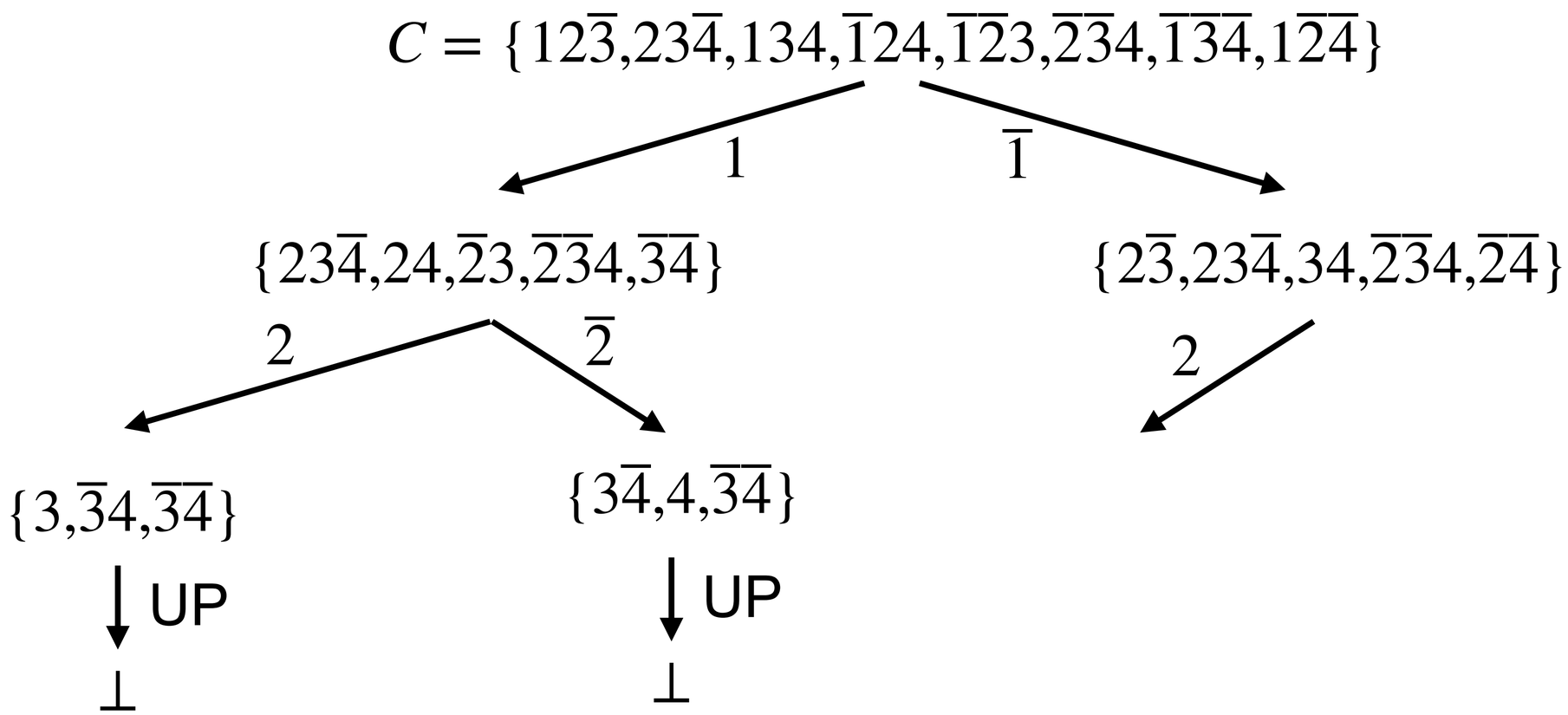

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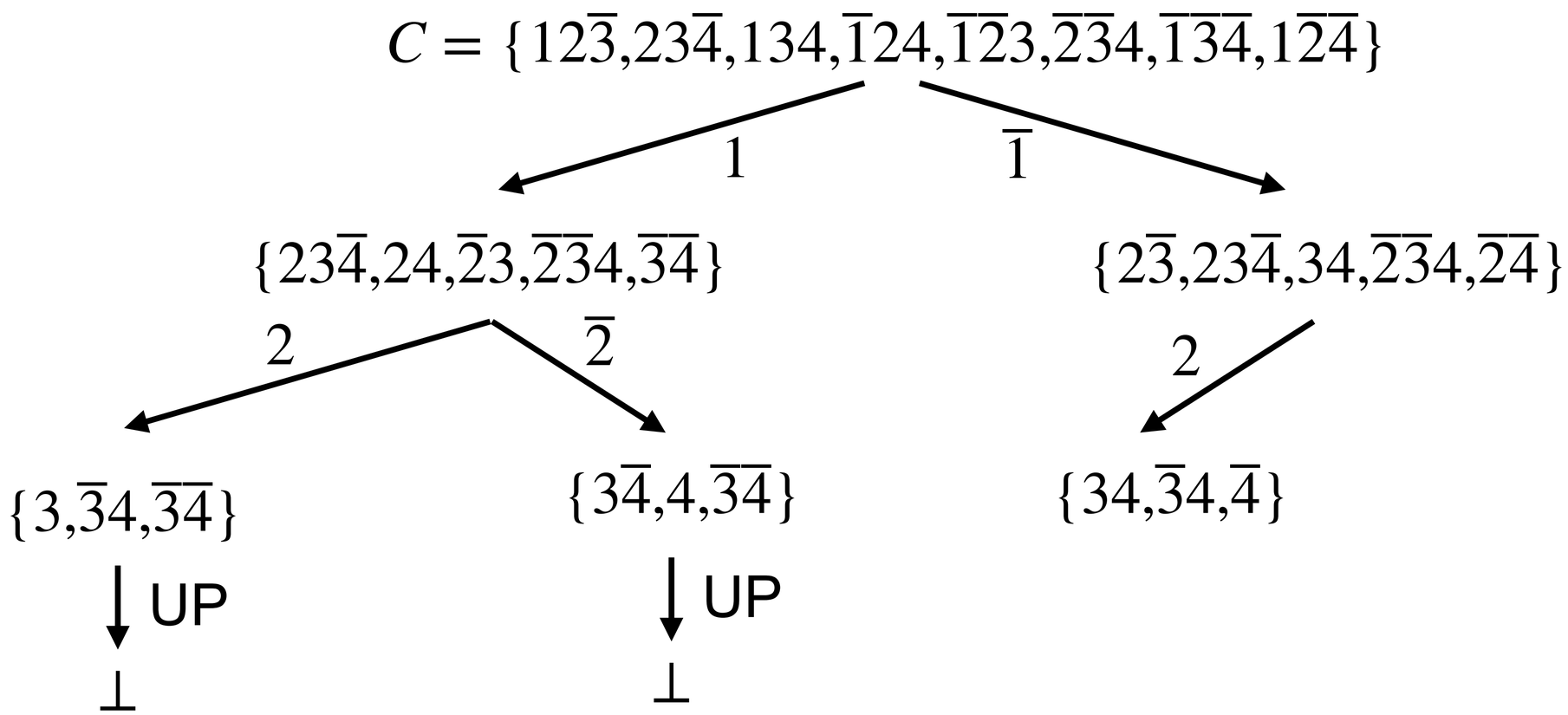

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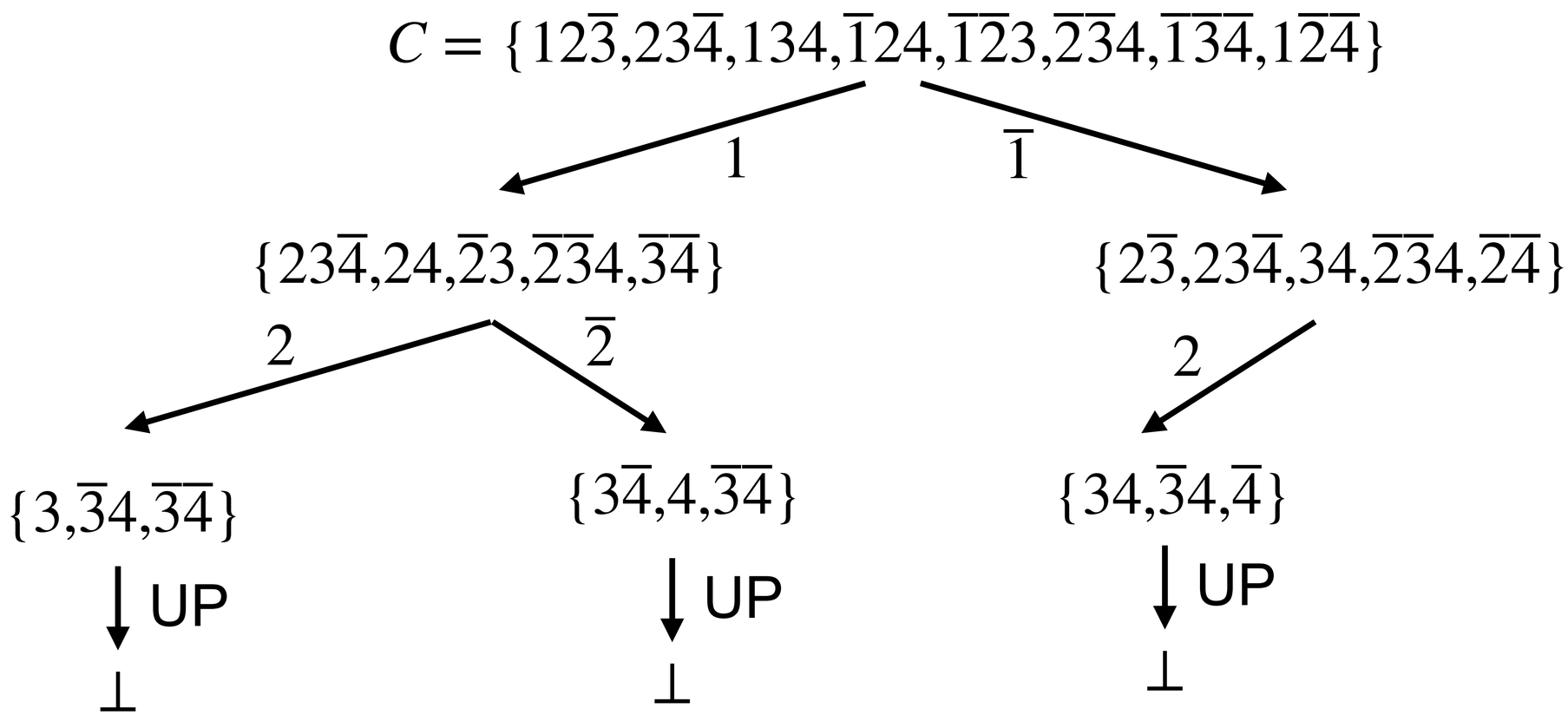

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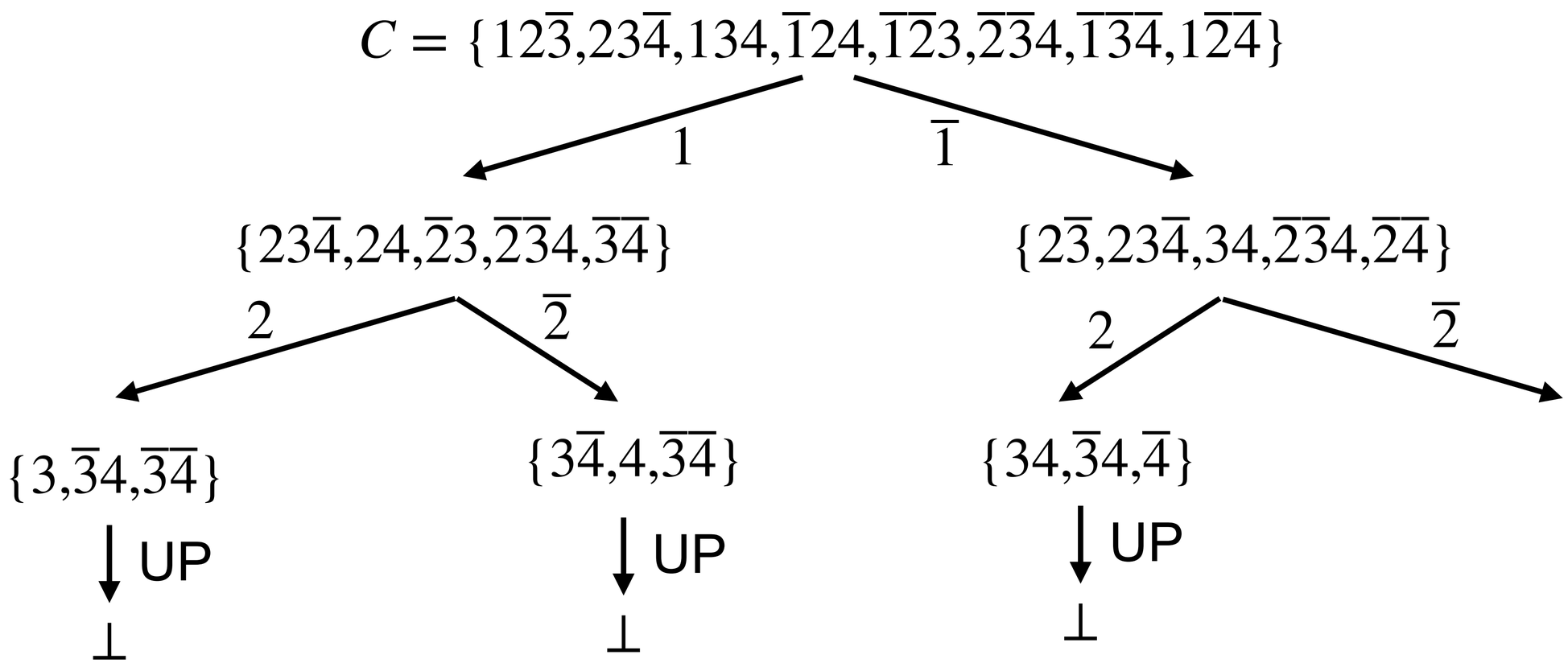

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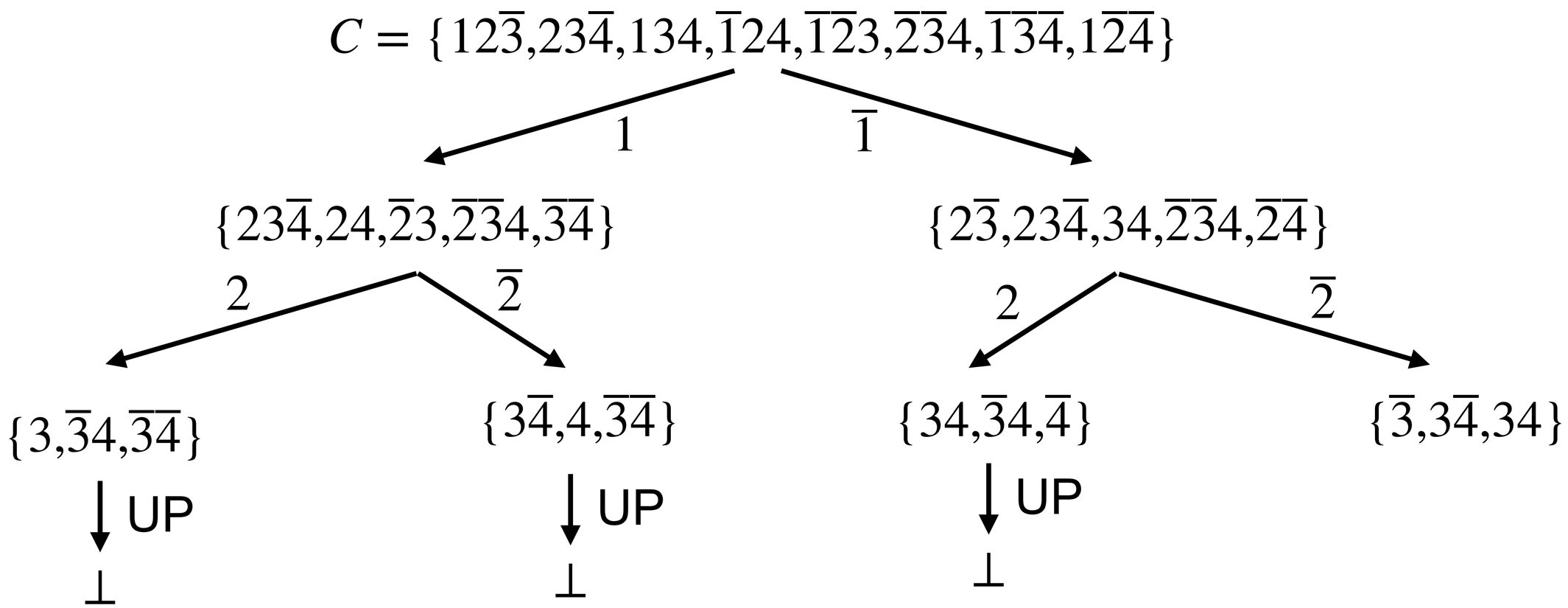

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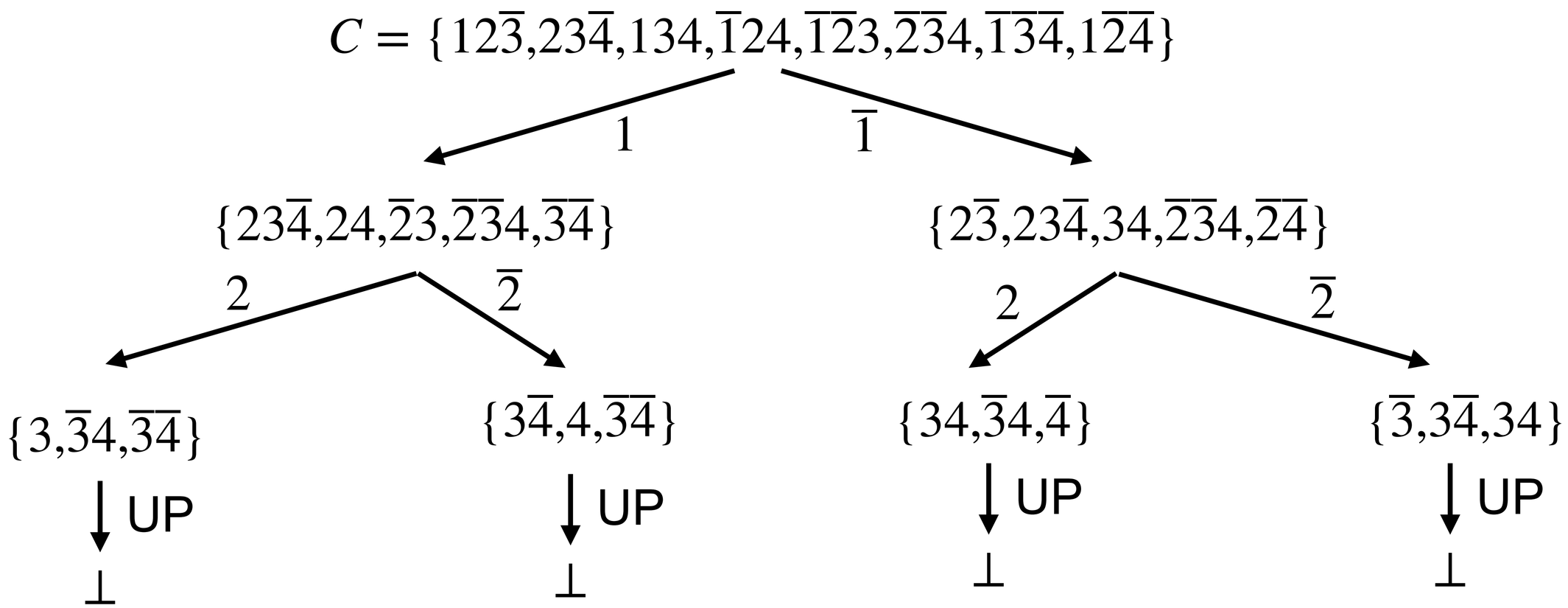

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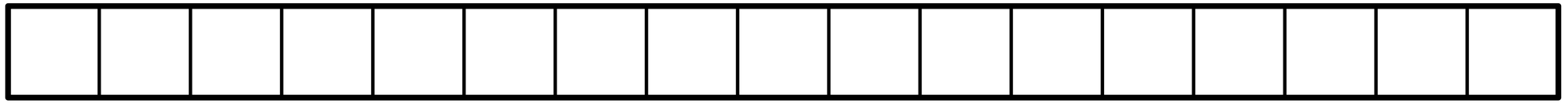

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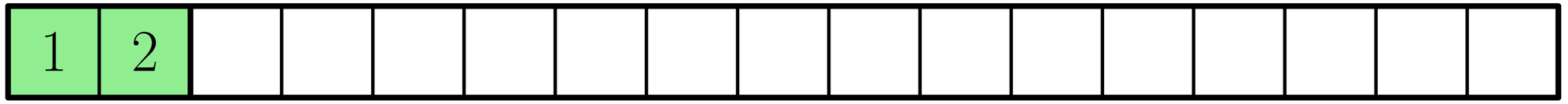

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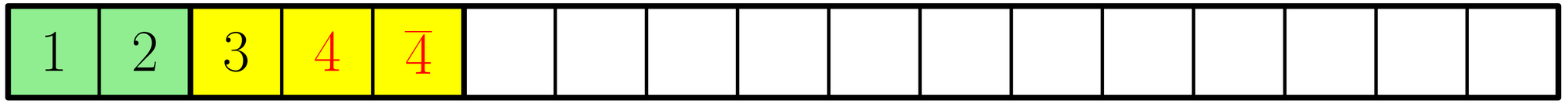

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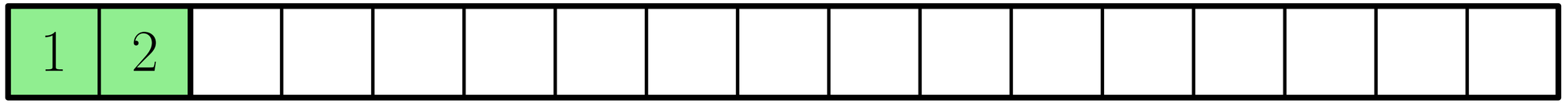

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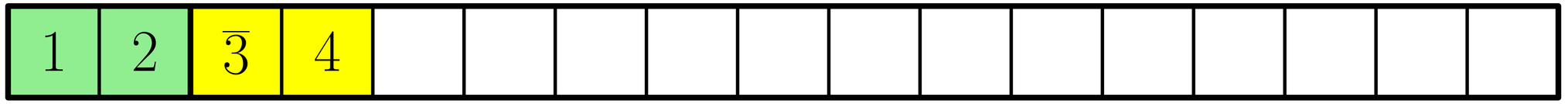

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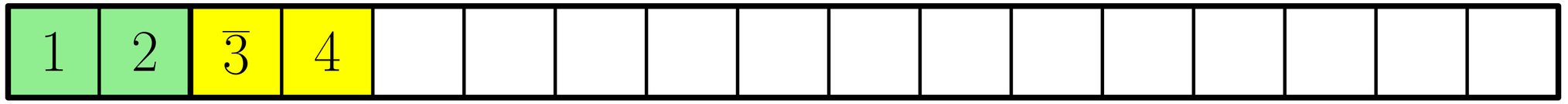

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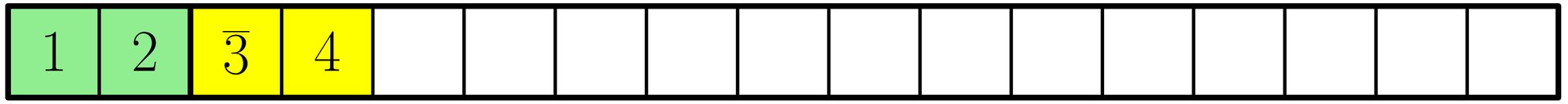

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Subdivided into *decision levels* by *decisions* (assignments not based on propagation)

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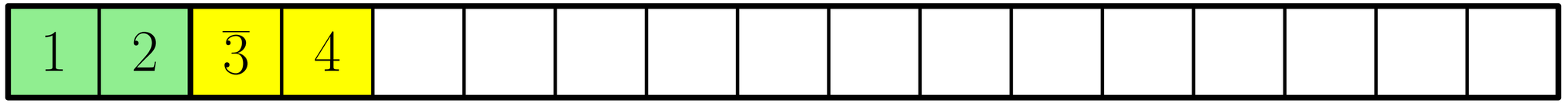

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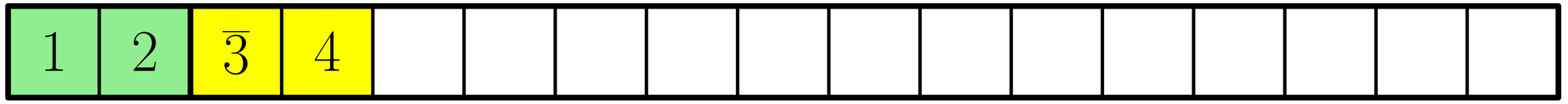

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Every other level starts on a decision

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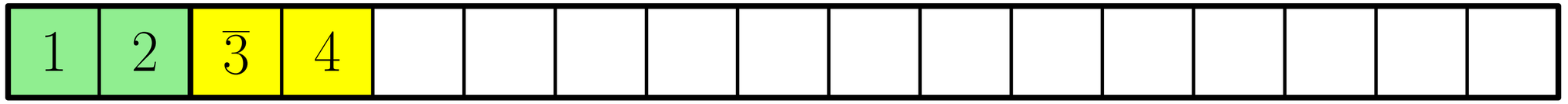

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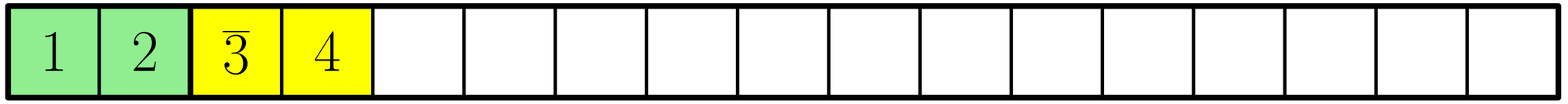

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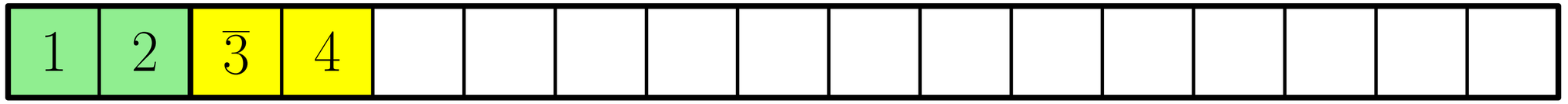

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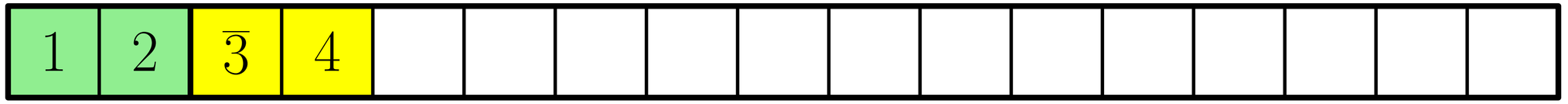

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- **Have to:** remove decision level from trail; what needs to happen to watched literals?

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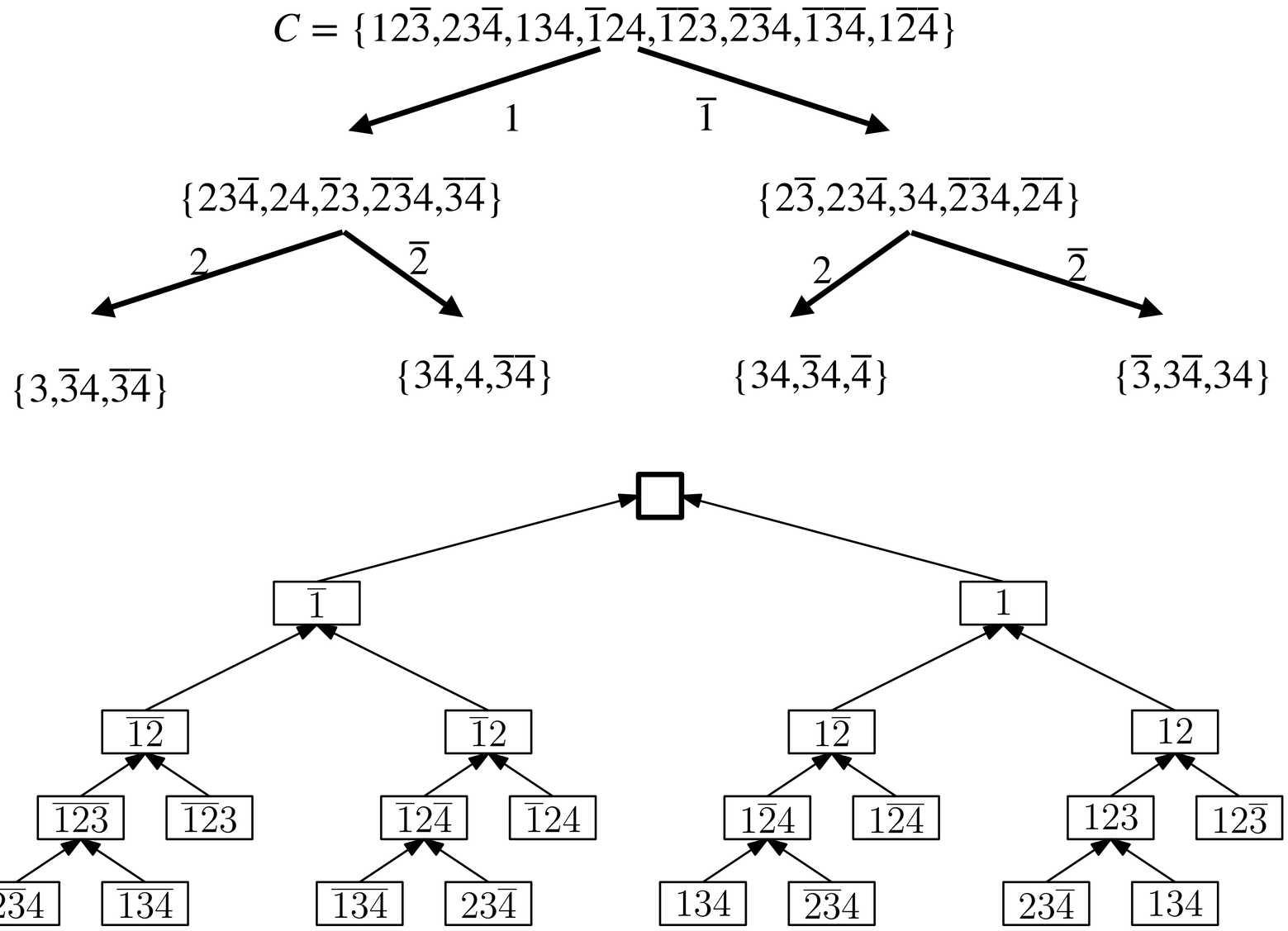
- More general: resolution proof of A (if we can get to a sink clause A)

Unsatisfiability Proof Analogue to DPLL Search Tree



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- These correspond to distances 1, 2, 3, 4; $\overline{1}$ for symmetry breaking.

Start of CDCL-Trail for Example

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0	$\overline{1}$	$\overline{1}$

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1	$\overline{2}$	Δ
1	3	123

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2	$\overline{4}$	Δ
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Idea:

- Resolve reasons for conflict variable $\rightarrow L$
- Resolve with reason for latest literal in L
- Until only one from level d remains in L

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Necessary: at most one literal at level d.

Idea:

- Resolve reasons for conflict variable $\rightarrow L$
- Resolve with reason for latest literal in L
- Until only one from level d remains in L

Example:

$$258 \diamond_8 \overline{678} = 25\overline{67},\ 25\overline{67} \diamond_5 \overline{357} = 2\overline{367},$$
$$2\overline{367} \diamond_7 147 = 12\overline{3}4\overline{6},\ 12\overline{3}4\overline{6} \diamond_6 246 = 12\overline{3}4.$$

$12\overline{3}4$ has only one literal from level 2!

Here, 24 is an (obviously) better choice.

Continuation of Example

level	literal	reason
0	$\overline{1}$	$\overline{1}$
1	$\overline{2}$	Δ
1	3	123
1	4	24

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level	literal	reason
0	$\overline{1}$	$\overline{1}$
1	$\overline{2}$	Δ
1	3	123
1	4	24
1	$\overline{5}$	$\overline{345}$
1	9	159
1	8	258
1	$\overline{6}$	$\overline{468}$
1	7	567
1	$\overline{9}$	$\overline{789}$

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0	$\overline{1}$	$\overline{1}$
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1	3	123
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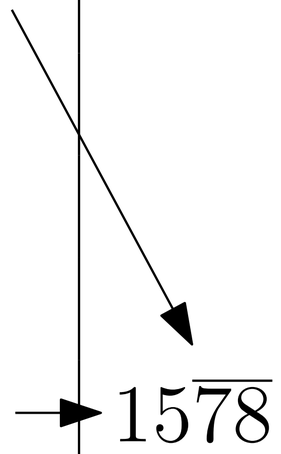

Continuation of Example

level	literal	reason	
0	$\overline{1}$	$\overline{1}$	
1	$\overline{2}$	Δ	
1	3	123	$\rightarrow 12$
1	4	24	$\rightarrow 12\overline{3}$
1	$\overline{5}$	$\overline{345}$	$\rightarrow 12\overline{34}$
1	9	159	
1	8	258	$\rightarrow 125\overline{4}$
1	$\overline{6}$	$\overline{468}$	$\rightarrow 15\overline{48}$
1	7	567	$\rightarrow 156\overline{8}$
1	$\overline{9}$	$\overline{789}$	$\rightarrow 15\overline{78}$

Continuation of Example

level	literal	reason
0	$\overline{1}$	$\overline{1}$
0	2	12
1	4	Δ
1	$\overline{3}$	$\overline{234}$
1	5	135
1	$\overline{8}$	$\overline{258}$
1	$\overline{6}$	$\overline{246}$
1	7	678
1	9	369
1	$\overline{9}$	$\overline{579}$

Continuation of Example

level	literal	reason	
0	$\overline{1}$	$\overline{1}$	
0	2	12	
1	4	Δ	
1	$\overline{3}$	$\overline{234}$	$\rightarrow 1\overline{24}$
1	5	135	$\rightarrow 13\overline{24}$
1	$\overline{8}$	$\overline{258}$	$\rightarrow 3\overline{245}$
1	$\overline{6}$	$\overline{246}$	$\rightarrow 38\overline{245}$
1	7	678	$\rightarrow 368\overline{5}$
1	9	369	
1	$\overline{9}$	$\overline{579}$	$\rightarrow 36\overline{57}$

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1	9	159	145
1	6	456	$45\overline{9}$
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1	$\overline{3}$	$\overline{369}$	$45\overline{69}$

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0	$\overline{1}$	$\overline{1}$	$\square$
0	2	12	1
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0	7	147	14
0	5	145	$14\overline{7}$
0	$\overline{3}$	$\overline{357}$	$\overline{57}$
0	$\overline{6}$	$\overline{567}$	$3\overline{57}$
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  - We usually take the first UIP we encounter (walking back up the trail)

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It is also possible that the learnt clause does not contain any decision variable!

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That implies termination!

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- Local Search
- Extensions (Incrementality, Cardinality SAT, XOR clauses, …)

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- Alternatives: VMTF, other conflict-based scores

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- Modern solvers: occasional rephasing (heuristics, longest conflict-free trail, random, …)

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- Activity-based (EVSIDS-like, clause score bumped for reasons in conflict analysis)

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- Other technique: scan for $x, y \in O$ with $x \to y$; then remove x

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when to run, how long to run, what candidates to run on, implementation details, …

Simple Pre- and Inprocessing

Subsumed Clause Detection:

A clause A is **subsumed** if it contains another clause $B \subsetneq A$.

A is a strictly weaker constraint than B and can be dropped.

A clause $A \supseteq \{x, \bar{x}\}$ is a *tautology* and can also be dropped.

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Bounded Variable Elimination (BVE): the single most effective preprocessing step.
Let φ consist of clauses C; let $S_+(x) = \{A \in C \mid x \in A\}$, $S\_(x) = \{B \in C \mid \bar{x} \in B\}$ and let $R_\pm(x) = \{A \diamond_x B \mid A \in S_+(x), B \in S\_(x)\}$ (drop tautologies).

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Then, $\mathrm{eliminate}(\varphi, x) := \varphi \setminus (S_+(x) \cup S\_(x)) \cup R_\pm(x)$ is satisfiable iff φ is.
BVE replaces φ by $\mathrm{eliminate}(\varphi, x)$ if it has at most some constant g more clauses.

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- Some information transfer to CDCL (beyond phases), e.g., score updates.

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- Many hyperparameters, optimizations and tweaks
- **Relevant for the exam:** CDCL, trail/decision levels, conflict resolution, high-level ideas (general idea of variable selection, existence of pre-/inprocessing, …)