Technische Universität Braunschweig
Institut für Betriebssysteme und Rechnerverbund
Algorithmik

Algorithm Engineering
Lecture 11: Decomposition

Problem Decomposition: Why?

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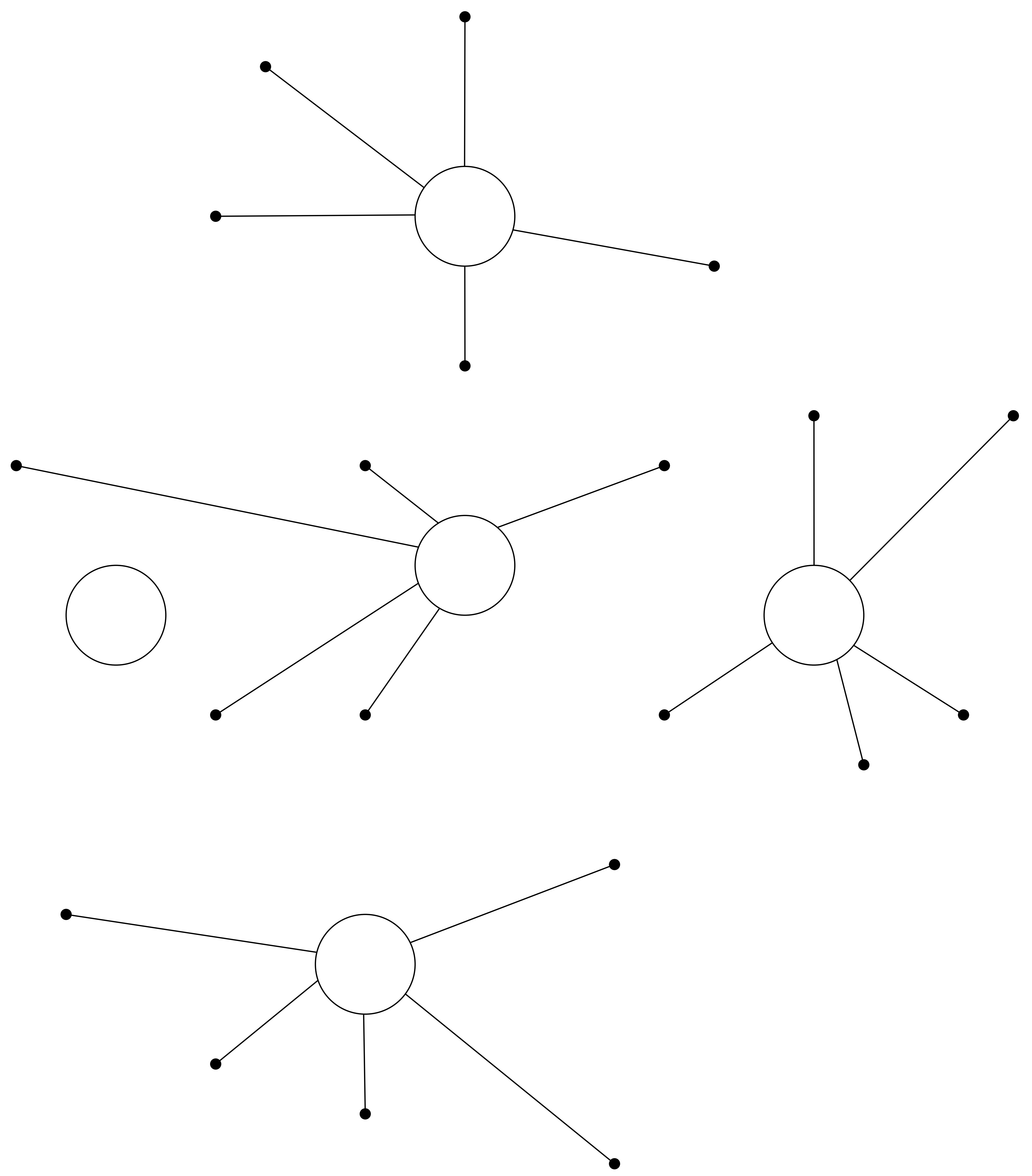
Some hard optimization problems:

- Composition of relatively loosely coupled individual problems
- Some components may already be hard
- **Or:** local decisions combined by a few global constraints
- **Idea:** Use that structure (*decompose* the problem)
- **This lecture:** how can we do that?

Example: Facility Location

Given:

- potential facility locations i
  - opening cost f_i
  - capacities c_i
- n clients j
  - every client must be assigned a facility
  - service costs s_{ij}
- at most c_i clients at facility i
- objective: minimize costs



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Now: three decomposition methods, three different decompositions.

First Decomposition: Lagrangian Relaxation

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Eliminate coupling constraints:

In our case: $\forall$clients $j : \sum_i y_{ij} = 1$; without these, we can solve every facility on its own.

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Independent subproblem for each facility!

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Very efficiently solvable in practice; how?

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Interpretation of μ: prices for violating constraints; we aim to iteratively find μ to maximize our bound $L(\mu)$, essentially solving a *Lagrangian dual* problem.

Decomposable Lagrangian Relaxation: Loop

So the loop becomes:

- Initialize penalties μ (e.g., to 0 or the LP dual values of the on the full problem)
- Solve the subproblems for the fixed μ; result:
  - This gives a lower bound (optimal solution of the relaxation)
  - Make heuristic solution from relaxed solution (for an upper bound)
- Update the penalties μ (cut-off/projection to ensure non-negativity/non-positivity)
- If lower bound $\neq$ upper bound, repeat.
- **In general:** needs branching for completeness or some termination criterion.

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- This updating rule is also called *subgradient method*

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Alternative methods: deflected subgradient (also add in previous step direction), …

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Each locally feasible choice is a variable now!

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Refined idea: at the LP optimum of the restricted master problem (RMP), find the variable minimizing the reduced costs. If it has negative reduced costs, add it; otherwise, the LP solution is optimal.

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Finding this variable is typically done via solving a *pricing* (sub-)problem.

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Dual: $\max \sum_i \alpha_i + \sum_j \beta_j$ s.t.

$\forall$facilities i, patterns $p \in P_i$: $\alpha_i + \sum_{j \in p} \beta_j \leq s_i^p$; reduced costs: $s_i^p - \alpha_i - \sum_{j \in p} \beta_j$

Reduced costs: $s_i^p - \alpha_i - \sum_{j \in p} \beta_j$; we get α_i, β_j from solving the RMP to optimality.

Pricing problem for a fixed facility i, assuming disabled facility is already in the RMP:

$\min f_i - \alpha_i + \sum_j (s_{ij} - \beta_j) y_j$ s.t.

$$\sum_j y_j \leq c_i,$$

$$y_j \in \{0,1\}$$

Very easy to solve (greedy works!); essentially the same that we saw for Lagrange!

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Stabilization: add slack variables in the RMP (allow violating coupling constraints at a cost); often piecewise-linear penalties. Prevents the dual from jumping too much from a *stability center* (a dual vector encoded in the penalties); only move the center on sufficient bound improvement.

Details — Column Generation: Pricing and Stabilization

Practical issue: We solve the (dual of) the RMP in each iteration, adding constraints. Its solution can jump wildly and be far off the true dual optimum.

Early iterations: meaningless, often later useless columns

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Stopping rules: compute a lower bound on the LP (often get this from pricing for free).

Details — Branch & Price

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- Effective use is considered an art by some mathematical programming researchers

Third Decomposition Approach: Benders' Decomposition

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  - Add cutting planes from a subproblem to refine the cost estimate
    (e.g., what is the best value for covering all clients with a given set of open facilities)

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So: given a fixed set of open facilities, optimally solve the client assignment problem

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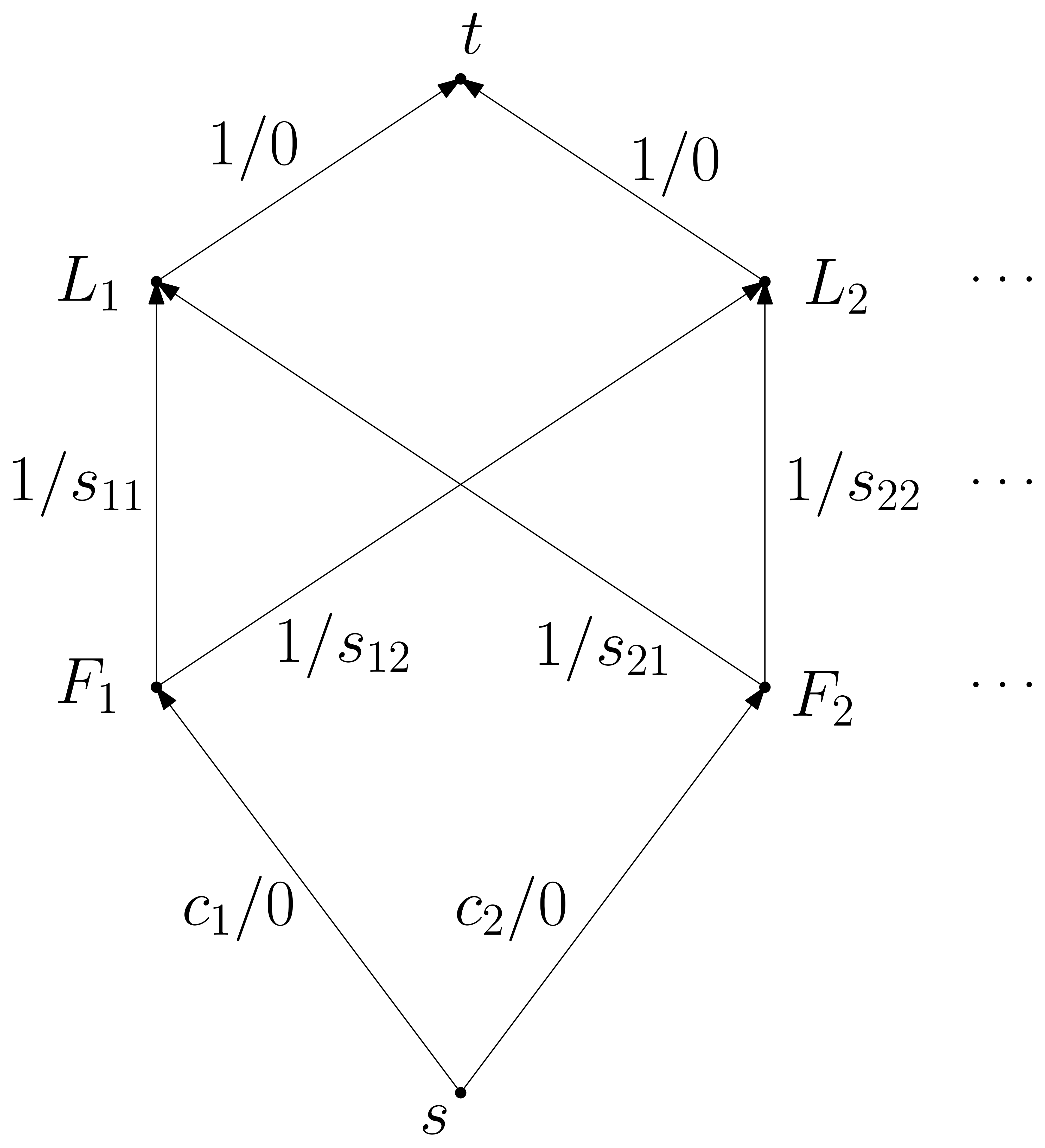

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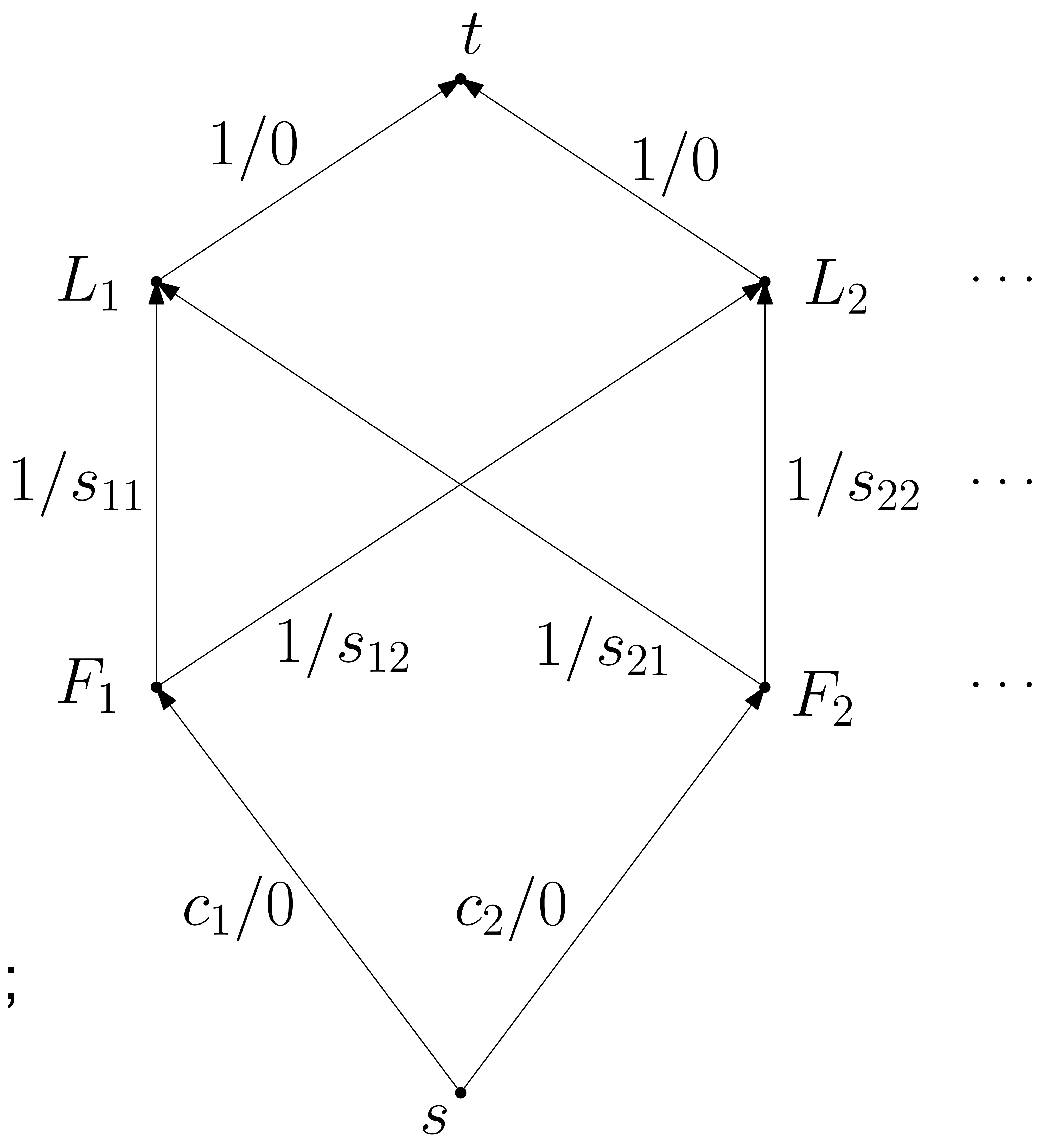

With fixed x_i, reduces to Min-Cost Max-s, t-Flow (in P); can be solved as LP and will automatically be integer.

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- If infeasible, we get a Farkas' lemma certificate (can be turned into a feasibility cut)

Details — Benders' Optimality Cuts

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Replacing the RHS by θ yields a (potentially violated) cut.
If it is not violated, we are done (proof via complementary slackness).

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But:

- There is no universal cut recipe (except for trivial no-goods that do not scale in practice)
- Problem-specific certificates from the subproblem needed (e.g., infeasible core, dual, …)

Conclusion

Seen decomposition ideas:

- Lagrangian relaxation (dualize away coupling constraints)
- Dantzig-Wolfe & Column Generation idea (solutions to local subproblems → variables)
- Benders' decomposition (approx. objective in master refined by cuts, LP subproblem)
- Logic-based Benders' decomposition (more general subproblem, inferred cuts)

General note: such approaches often require a lot of tuning to reach their full potential.

So, before a serious implementation attempt, look up the details to keep in mind!